

# Local Fractional Laplace Variational Iteration Method for Solving Nonlinear Partial Differential Equations on Cantor Sets within Local Fractional Operators



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## Abstract:

**In this work, we discuss solutions of the nonlinear partial differential equations on Cantor sets within local fractional operators. The nondifferentiable approximate solutions are obtained by using the local fractional Laplace variational iteration method, which is the coupling method of local fractional variational iteration method and Laplace transform. The obtained results show the efficiency and accuracy of implements of the present method.**

**Keywords:** Nonlinear differential equation; Fractional Gas dynamics equation; Cantor set; Yang-Laplace transform; local fractional variational iteration method.

## 1. Introduction

Local fractional calculus has played an important role in areas ranging from fundamental science to engineering in the past ten years [1-3] has been applied to a wide class of complex problems encompassing physics, biology, mechanics, and interdisciplinary areas [4-9]. Various methods, for example, the Adomian decomposition method [10], the Adomian-Meyers modified decomposition method [11], the variational iteration method [12], the homotopy perturbation method [13, 14], the fractal Laplace and Fourier transforms [15], the homotopy analysis method [16], the

heat balance integral method [17-19], the fractional variational iteration method [20-22], the fractional subequation method [23, 24], have been utilized to solve fractional differential equations [3, 15].

There is a long-standing interest in extending the classical calculus to non integer orders [25-27] because the applications of fractional calculus (integrals and derivatives of any real or complex order) have attracted a great deal of attention in recent years. For example, local fractional differential equations are increasingly used to model many problems in biology, chemistry,

economic, engineering, physics and other areas of applications. The local fractional differential equations have become a useful tool for describing nonlinear phenomena of science and engineering models. Several authors including Beyer and Kempfle [28], Schneider and Wyss [29], Mainardi [30], Huang and Liu [31], He [32, 33], Faraz et al. [34] discussed some methods and solutions of fractional differential equations.

The purpose of this paper is to introduce a new method for local fractional partial differential equations. Our aim is to extend the application of the proposed method to obtain the analytical solutions to Gas dynamics and Burger's equations on Cantor sets.

The paper is organized as follows: **In Section 2** we introduce the notions and the notations of local fractional calculus theory used in this paper. **In Section 3** we give the local fractional Laplace variational iteration method. **Section 4** presents the solutions for some nonlinear partial differential equations in Cantor set conditions. **Section 5** is devoted to our conclusions.

## 2. Mathematical Fundamentals

### 2.1 Local fractional calculus [35-37,40]

**Definition 1.** Suppose that there is the relation

$$|f(x) - f(x_0)| < \varepsilon^\alpha, 0 < \alpha \leq 1 \quad (2.1)$$

with  $|x - x_0| < \delta$ , for  $\varepsilon, \delta > 0$  and  $\varepsilon, \delta \in R$ , then the function  $f(x)$  is called local fractional continuous at  $x = x_0$  and it is denoted by  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

**Definition 2.** Suppose that the function  $f(x)$  satisfies condition (2.1), for  $x \in (a, b)$ ; it is so

called local fractional continuous on the interval  $(a, b)$ , denoted by  $f(x) \in C_\alpha(a, b)$ .

**Definition 3.** In fractal space, let  $f(x) \in C_\alpha(a, b)$ , local fractional derivative of  $f(x)$  of order  $\alpha$  at  $x = x_0$  is given by

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x) \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha} \quad (2.2)$$

where  $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$ .

The formulas of local fractional derivatives of special functions [35] used in the paper are as follows:

$$D_x^{(\alpha)} a g(x) = a D_x^{(\alpha)} g(x), \quad (2.3)$$

$$\frac{d^\alpha}{dx^\alpha} \left( \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad n \in N \quad (2.4)$$

**Definition 4.** A partition of the interval  $[a, b]$  is denoted as  $(t_j, t_{j+1})$ ,  $j = 0, \dots, N-1$ ,  $t_0 = a$  and  $t_N = b$  with  $\Delta t_j = t_{j+1} - t_j$  and  $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$  Local fractional integral of  $f(x)$  in the interval  $[a, b]$  is given by

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha. \quad (2.5)$$

The formulas of local fractional integrals of special functions [35] used in the paper are as follows:

$${}_0 I_x^{(\alpha)} a g(x) = a {}_0 I_x^{(\alpha)} g(x), \quad (2.6)$$

$${}_0 I_t^{(\alpha)} \left( \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}, \quad n \in N \quad (2.7)$$

**Definition 5.** In fractal space, the Mittag-Leffler function, sine function and cosine function are defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad 0 < \alpha \leq 1$$

(2.8)

$$\sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}, \quad 0 < \alpha \leq 1$$

(2.9)

$$\cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]}, \quad 0 < \alpha \leq 1$$

(2.10)

## 2.2 Local fractional Laplace Transform and Its Inverse Formula\_ see [35-37].

**Definition 6.** Let

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty |f(x)|(dx)^\alpha < k < \infty.$$

The Yang-Laplace transforms of  $f(x)$  is given by

$$L_\alpha\{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \quad 0 < \alpha \leq 1$$

(2.11)

where the latter integral converges and  $s^\alpha \in R^\alpha$ .

**Definition 7.** The inverse formula of the Yang-Laplace transforms of  $f(x)$  is given by

$$L_\alpha^{-1}\{f_s^{L,\alpha}(s)\} = f(t) = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s) (ds)^\alpha, \quad 0 < \alpha \leq 1$$

(2.12)

where  $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$ ; fractal imaginary unit  $i^\alpha$  and  $\text{Re}(s) = \beta > 0$ .

## 2.3 Some Basic Properties of Local Fractional Laplace Transform [35-36]).

Let  $L_\alpha\{f(x)\} = f_s^{L,\alpha}(s)$  and  $L_\alpha\{g(x)\} = g_s^{L,\alpha}(s)$ , then we have the following formulas

$$L_\alpha\{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s)$$

(2.13)

$$L_\alpha\{E_\alpha(c^\alpha x^\alpha) f(x)\} = f_s^{L,\alpha}(s-c)$$

(2.14)

$$L_\alpha\{f^{(k\alpha)}(x)\} = s^{k\alpha} f_s^{L,\alpha}(s) - s^{(k-1)\alpha} f(0) - s^{(k-2)\alpha} f^{(\alpha)}(0) - \dots - f^{((k-1)\alpha)}(0)$$

(2.15)

$$L_\alpha\{E_\alpha(a^\alpha x^\alpha)\} = \frac{1}{s^\alpha - a^\alpha}$$

(2.16)

$$L_\alpha\left\{\frac{1}{\Gamma(1+\alpha)} \int_0^x f(t) (dt)^\alpha\right\} = \frac{f_s^{L,\alpha}(s)}{s^\alpha}$$

(2.17)

$$L_\alpha\{\sin_\alpha(a^\alpha x^\alpha)\} = \frac{a^\alpha}{s^{2\alpha} + a^{2\alpha}}$$

(2.18)

$$L_\alpha\{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}}$$

(2.19)

## 3. Local Fractional Laplace Variational Iteration Method.

We consider a general nonlinear local fractional partial differential equation:

$$L_\alpha u(x,t) + R_\alpha u(x,t) + N_\alpha u(x,t) = f(x,t), \quad t > 0, x \in R, 0 < \alpha \leq 1$$

(3.1)

where  $L_\alpha$  is the linear local fractional operator,  $R_\alpha$  is a linear local fractional operator of order less than  $L_\alpha$ ,  $N_\alpha$  represents the general nonlinear differential operator, and  $f(x,t)$  is the source term.

According to the rule of local fractional variational iteration method, the correction local fractional functional for Eq. (3.1) is constructed as [42-45]:

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)} \left( \frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)} [L_\alpha u_n(t) + R_\alpha \tilde{u}_n(t) + N_\alpha u_n(t) - f(t)] \right) \quad (3.2)$$

where  $\frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)}$  is a fractal Lagrange and  $L_\alpha$  in Eq. (3.1) are  $k\alpha$  times local fractional partial differential equations.

For initial value problems of Eq. (3.1), we can start with:

$$u_0(x) = u(0) + \frac{x^\alpha}{\Gamma(1+\alpha)} u^{(\alpha)}(0) + \dots + \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} u^{((k-1)\alpha)}(0) \quad (3.3)$$

We now take Yang-Laplace transform of Eq. (3.2), namely:

$$\mathcal{E}_\alpha \{u_{n+1}(x)\} = \mathcal{E}_\alpha \{u_n(x)\} + \mathcal{E}_\alpha \left\{ {}_0I_x^{(\alpha)} \left( \frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)} [L_\alpha u_n(t) + R_\alpha u_n(t) + N_\alpha \tilde{u}_n(t) - f(x,t)] \right) \right\} \quad (3.4)$$

Or

$$\mathcal{E}_\alpha \{u_{n+1}(x)\} = \mathcal{E}_\alpha \{u_n(x)\} + \mathcal{E}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \mathcal{E}_\alpha \{L_\alpha u_n(x) + R_\alpha u_n(x) + N_\alpha \tilde{u}_n(x) - f(x,t)\} \right\} \quad (3.5)$$

Taking the local fractional variation of (3.5), which is given by:

$$\delta^\alpha (\mathcal{E}_\alpha \{u_{n+1}(x)\}) = \delta^\alpha (\mathcal{E}_\alpha \{u_n(x)\}) + \delta^\alpha \left( \mathcal{E}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \mathcal{E}_\alpha \{L_\alpha u_n(x) + R_\alpha u_n(x) + N_\alpha \tilde{u}_n(x) - f(x,t)\} \right\} \right) \quad (3.6)$$

By using computation of (3.6), we get:

$$\delta^\alpha (\mathcal{E}_\alpha \{u_{n+1}(x)\}) = \delta^\alpha (\mathcal{E}_\alpha \{u_n(x)\}) + \mathcal{E}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha (\mathcal{E}_\alpha \{L_\alpha u_n(x)\}) \right\} \quad (3.7)$$

Hence, from (3.7) we get:

$$1 + \mathcal{E}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} s^{k\alpha} = 0, \quad (3.8)$$

where

$$\begin{aligned} & \delta^\alpha (\mathcal{E}_\alpha \{L_\alpha u_n(x)\}) \\ &= \delta^\alpha \left( s^{k\alpha} \mathcal{E}_\alpha \{u_n(x)\} - s^{(k-1)\alpha} u_n(0) - \dots - u_n^{((k-1)\alpha)}(0) \right) \\ &= s^{k\alpha} \delta^\alpha (\mathcal{E}_\alpha \{u_n(x)\}) \end{aligned} \quad (3.9)$$

Therefore, we get

$$\mathcal{E}_\alpha \left\{ \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \right\} = -\frac{1}{s^{k\alpha}} \quad (3.10)$$

Taking the inverse version of the Yang-Laplace transform, we have:

$$\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} = \mathcal{E}_\alpha^{-1} \left\{ -\frac{1}{s^{k\alpha}} \right\} = -\frac{(x)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}, \quad k \in \mathbb{N} \quad (3.11)$$

In view of Eq. (3.11), we obtain:

$$\begin{aligned} \mathcal{E}_\alpha \{u_{n+1}(x)\} &= \mathcal{E}_\alpha \{u_n(x)\} \\ &- \mathcal{E}_\alpha \left\{ {}_0I_x^{(\alpha)} \left( \frac{(x-t)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} [L_\alpha \tilde{u}_n(t) + R_\alpha u_n(t) + N_\alpha u_n(t) - f(x,t)] \right) \right\} \end{aligned} \quad (3.12)$$

Therefore, we have the following iteration algorithm:

$$\begin{aligned} \mathcal{E}_\alpha \{u_{n+1}(x)\} &= \mathcal{E}_\alpha \{u_n(x)\} - \mathcal{E}_\alpha \left\{ \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \right\} \\ &\mathcal{E}_\alpha \{L_\alpha u_n(x) + R_\alpha u_n(x) + N_\alpha \tilde{u}_n(x) - f(x,t)\} \end{aligned} \quad (3.13)$$

Or

$$\begin{aligned} \mathcal{E}_\alpha \{u_{n+1}(x)\} &= \mathcal{E}_\alpha \{u_n(x)\} - \\ &\frac{1}{s^{k\alpha}} \mathcal{E}_\alpha \{L_\alpha u_n(x) + R_\alpha u_n(x) + N_\alpha \tilde{u}_n(x) - f(x,t)\} \end{aligned} \quad (3.14)$$

where the initial value reads as

$$\begin{aligned} u_0(x) &= \mathcal{E}_\alpha^{-1} \left( \frac{s^{(k-1)\alpha} u(0) + s^{(k-2)\alpha} u^{(\alpha)}(0) + \dots + u_n^{((k-1)\alpha)}(0)}{s^{k\alpha}} \right) \\ &= u(0) + \frac{x^\alpha}{\Gamma(1+\alpha)} u^{(\alpha)}(0) + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} u^{(2\alpha)}(0) \\ &\quad + \dots + \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} u^{((k-1)\alpha)}(0) \end{aligned} \quad (3.15)$$

Thus, the local fractional series solution of Eq. (3.1) is

$$u = \lim_{n \rightarrow \infty} E_\alpha^{-1} \{ E_\alpha \{ u_n \} \} \quad (3.16)$$

#### 4. Applications to Nonlinear Partial Differential equations on Cantor Sets.

In this section, tow examples for nonlinear partial differential equations on Cantor sets will demonstrate the efficiency of local fractional Laplace variational iteration method (LFLVIM).

**Example 1.** Let us consider the nonlinear partial differential equation on cantor set[26]:

$$\frac{\partial^{2\alpha} u(x, y)}{\partial x^{2\alpha}} + u^2(x, y) - \left( \frac{\partial^\alpha u(x, y)}{\partial y^\alpha} \right)^2 = 0, \quad 0 < \alpha \leq 1 \quad (4.1)$$

with the initial value conditions as follows:

$$u(0, y) = 0, \quad \frac{\partial^\alpha u(0, y)}{\partial x^\alpha} = E_\alpha(y^\alpha) \quad (4.2)$$

Using relation (3.13) we structure the iterative relation as:

$$\begin{aligned} E_\alpha \{ u_{n+1}(x, y) \} &= E_\alpha \{ u_n(x, y) \} - E_\alpha \left\{ \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\ &* E_\alpha \left\{ \frac{\partial^{2\alpha} u_n(x, y)}{\partial x^{2\alpha}} + u_n^2(x, y) - \left( \frac{\partial^\alpha u_n(x, y)}{\partial y^\alpha} \right)^2 \right\} \\ &= E_\alpha \{ u_n(x, y) \} - \frac{1}{s^{2\alpha}} * \\ &\left( s^{2\alpha} E_\alpha \{ u_n(x, y) \} - s^\alpha u_n(0, y) \right. \\ &\left. - u_n^{(\alpha)}(0, y) + E_\alpha \{ u_n^2(x, y) \} - E_\alpha \left\{ \left( \frac{\partial^\alpha u_n(x, y)}{\partial y^\alpha} \right)^2 \right\} \right) \\ &= \frac{1}{s^\alpha} u_n(0, y) + \frac{1}{s^{2\alpha}} u_n^{(\alpha)}(0, y) - \frac{1}{s^{2\alpha}} E_\alpha \{ u_n^2(x, y) \} \\ &+ \frac{1}{s^{2\alpha}} E_\alpha \left\{ \left( \frac{\partial^\alpha u(x, y)}{\partial y^\alpha} \right)^2 \right\} \end{aligned} \quad (4.3)$$

In view of Eq. (3.15), the initial value reads:

$$\begin{aligned} u_0(x, y) &= u(0, y) + \frac{x^\alpha}{\Gamma(1+\alpha)} u^{(\alpha)}(0, y) \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(y^\alpha) \end{aligned} \quad (4.4)$$

Hence, we get the first approximation, namely:

$$\begin{aligned} E_\alpha \{ u_1(x, y) \} &= \frac{1}{s^\alpha} u_0(0, y) + \frac{1}{s^{2\alpha}} u_0^{(\alpha)}(0, y) \\ &- \frac{1}{s^{2\alpha}} E_\alpha \{ u_0^2(x, y) \} + \frac{1}{s^{2\alpha}} E_\alpha \left\{ \left( \frac{\partial^\alpha u_0(x, y)}{\partial y^\alpha} \right)^2 \right\} \\ &= \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \end{aligned} \quad (4.5)$$

Thus,

$$u_1(x, y) = E_\alpha^{-1} \left( \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \right) = \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(y^\alpha) \quad (4.6)$$

The second approximations reads:

$$\begin{aligned} E_\alpha \{ u_2(x, y) \} &= \frac{1}{s^\alpha} u_1(0, y) + \frac{1}{s^{2\alpha}} u_1^{(\alpha)}(0, y) \\ &- \frac{1}{s^{2\alpha}} E_\alpha \{ u_1^2(x, y) \} + \frac{1}{s^{2\alpha}} E_\alpha \left\{ \left( \frac{\partial^\alpha u_1(x, y)}{\partial y^\alpha} \right)^2 \right\} \\ &= \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \end{aligned} \quad (4.7)$$

Therefore, we get

$$u_2(x, y) = E_\alpha^{-1} \left( \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \right) = \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(y^\alpha) \quad (4.8)$$

The other approximations are written as:

$$\begin{aligned} E_\alpha \{ u_3(x, y) \} &= \frac{1}{s^\alpha} u_2(0, y) + \frac{1}{s^{2\alpha}} u_2^{(\alpha)}(0, y) \\ &- \frac{1}{s^{2\alpha}} E_\alpha \{ u_2^2(x, y) \} + \frac{1}{s^{2\alpha}} E_\alpha \left\{ \left( \frac{\partial^\alpha u_2(x, y)}{\partial y^\alpha} \right)^2 \right\} \\ &= \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \end{aligned} \quad (4.9)$$

Therefore, we get

$$u_3(x, y) = E_\alpha^{-1} \left( \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \right) = \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(y^\alpha), \quad (4.10)$$

and so on.

Consequently, the local fractional series solution is:

$$u(x, y) = \lim_{n \rightarrow \infty} E_\alpha^{-1} \{ E_\alpha \{ u_n(x, y) \} \} = \lim_{n \rightarrow \infty} E_\alpha^{-1} \left( \frac{1}{s^{2\alpha}} E_\alpha(y^\alpha) \right) = \frac{x^\alpha}{\Gamma(1+\alpha)} E_\alpha(y^\alpha) \quad (4.11)$$

**Example 2.** Let us consider the following Gas dynamics equation on cantor set [26]

$$\frac{\partial^\alpha u(x, y)}{\partial x^\alpha} + \frac{1}{2} \frac{\partial^\alpha u^2(x, y)}{\partial y^\alpha} - u(x, y)(1-u(x, y)) = 0, \quad (4.12)$$

with the initial value conditions as follows:

$$u(0, y) = E_\alpha(-y^\alpha) \quad (4.13)$$

Using relation (3.13) we structure the iterative relation as:

$$\begin{aligned} E_\alpha \{ u_{n+1}(x, y) \} &= E_\alpha \{ u_n(x, y) \} - E_\alpha \{ 1 \} \\ &+ E_\alpha \left\{ \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} + \frac{1}{2} \frac{\partial^\alpha u^2(x, y)}{\partial y^\alpha} - u(x, y)(1-u(x, y)) \right\} \\ &= E_\alpha \{ u_n(x, y) \} - \frac{1}{s^\alpha} \left( \frac{s^\alpha E_\alpha \{ u_n(x, y) \} - u_n(0, y) +}{2} \frac{\partial^\alpha E_\alpha \{ u_n(x, y) \}}{\partial y^\alpha} - u_n(1-u_n) \right) \\ &= \frac{1}{s^\alpha} u_n(0, y) - \frac{1}{2} \frac{1}{s^\alpha} E_\alpha \left\{ \frac{\partial^\alpha u_n^2(x, y)}{\partial y^\alpha} \right\} \\ &+ \frac{1}{s^\alpha} E_\alpha \{ u_n(x, y)(1-u_n(x, y)) \} \end{aligned} \quad (4.14)$$

In view of Eq. (3.15), the initial value reads:

$$u_0(x, y) = u(0, y) = E_\alpha(-y^\alpha) \quad (4.15)$$

Hence, we get the first approximation, namely:

$$\begin{aligned} E_\alpha \{ u_1(x, y) \} &= \frac{1}{s^\alpha} u_0(0, y) - \frac{1}{2} \frac{1}{s^\alpha} E_\alpha \left\{ \frac{\partial^\alpha u_0^2(x, y)}{\partial y^\alpha} \right\} \\ &+ \frac{1}{s^\alpha} E_\alpha \{ u_0(x, y)(1-u_0(x, y)) \} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{s^\alpha} E_\alpha(-y^\alpha) + \frac{1}{s^{2\alpha}} E_\alpha(-2y^\alpha) \\ &+ \frac{1}{s^{2\alpha}} E_\alpha(-y^\alpha)(1-E(-y^\alpha)) \\ &= E_\alpha(-y^\alpha) \left( \frac{1}{s^\alpha} + \frac{1}{s^{2\alpha}} \right) \end{aligned} \quad (4.16)$$

Thus,

$$\begin{aligned} u_1(x, y) &= E_\alpha^{-1} \left( E_\alpha(-y^\alpha) \left( \frac{1}{s^\alpha} + \frac{1}{s^{2\alpha}} \right) \right) \\ &= E_\alpha(-y^\alpha) \left( 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} \right) \end{aligned} \quad (4.17)$$

The second approximations reads:

$$\begin{aligned} E_\alpha \{ u_2(x, y) \} &= \frac{1}{s^\alpha} u_1(0, y) - \frac{1}{2} \frac{1}{s^\alpha} E_\alpha \left\{ \frac{\partial^\alpha u_1^2(x, y)}{\partial y^\alpha} \right\} \\ &+ \frac{1}{s^\alpha} E_\alpha \{ u_1(x, y)(1-u_1(x, y)) \} \\ &= E_\alpha(-y^\alpha) \left( \frac{1}{s^\alpha} + \frac{1}{s^{2\alpha}} + \frac{1}{s^{3\alpha}} \right) \end{aligned} \quad (4.18)$$

Therefore, we get

$$\begin{aligned} u_2(x, y) &= E_\alpha^{-1} \left( E_\alpha(-y^\alpha) \left( \frac{1}{s^\alpha} + \frac{1}{s^{2\alpha}} + \frac{1}{s^{3\alpha}} \right) \right) \\ &= E_\alpha(-y^\alpha) \left( 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \right), \end{aligned} \quad (4.19)$$

and so on.

Consequently, the local fractional series solution is:

$$\begin{aligned} u(x, y) &= \lim_{n \rightarrow \infty} E_\alpha^{-1} \{ E_\alpha \{ u_n(x, y) \} \} \\ &= \lim_{n \rightarrow \infty} E_\alpha^{-1} \left( E_\alpha(-y^\alpha) \sum_{k=0}^n \frac{1}{s^{(k+1)\alpha}} \right) \\ &= \lim_{n \rightarrow \infty} \left( E_\alpha(-y^\alpha) \sum_{k=0}^n \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \right) \\ &= E_\alpha(-y^\alpha) E_\alpha(x^\alpha) = E_\alpha((x-y)^\alpha) \end{aligned} \quad (4.20)$$

## 5. Conclusions

The local fractional Laplace variational iteration method was applied to the nonlinear

partial differential equations defined on Cantor sets with the fractal conditions. The local fractional Laplace variational iteration method was proved to be effective and very reliable for analytic purposes. This technique

may be considered an alternative and efficient method for finding approximate solutions of both linear and nonlinear local fractional differential equations.

## 6. References

- [1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, (2006).
- [2] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, UK, (2010).
- [3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, (1999).
- [4] R. L. Magin, *Fractional Calculus in Bioengineering*, Begerll House, West Redding, Conn, USA, (2006).
- [5] J. Klafter, S. C. Lim, and R. Metzler, *Fractional Dynamics in Physics: Recent Advances*, World Scientific, Singapore, (2011).
- [6] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, (2008).
- [7] B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Springer, New York, NY, USA, (2003).
- [8] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, Germany, (2011).
- [9] J. A. Tenreiro Machado, A. C. J. Luo, and D. Baleanu, *Nonlinear Dynamics of Complex Systems: Applications in Physical, Biological and Financial Systems*, Springer, New York, NY, USA, (2011).
- [10] J. S. Duan, T. Chaolu, R. Rach, and L. Lu, "The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations," *Computers & Mathematics With Applications*, (2013).
- [11] J. S. Duan, T. Chaolu, and R. Rach, "Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8370–8392, (2012).
- [12] S. Das, "Analytical solution of a fractional diffusion equation by variational iteration method," *Computers & Mathematics with Appl.*, vol. 57, no. 3, pp. 483-487, (2009).
- [13] S. Momani and Z. Odibat, "Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 910–919, (2007).
- [14] S. Momani and Z. Odibat, "Homotopy perturbation method for nonlinear partial differential equations," *Physics Letters A*, vol. 365, no. 5-6, pp. 345-350, (2007).
- [15] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston, Mass, USA, (2012).
- [16] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, (2009).
- [17] J. Hristov, "Heat-balance integral to fractional (half-time) heat diffusion sub-model,

- ”Thermal Science, vol. 14, no. 2, pp.291–316, (2010).
- [18] J. Hristov, “Integral-balance solution to the Stokes’ first problem of a viscoelasticity generalized second grade fluid,” *Thermal Science*, vol. 16, no. 2, pp. 395–410, (2012).
- [19] J. Hristov, “Transient flow of a generalized second grade fluid due to a constant surface shear stress: an approximate integral balance solution,” *International Review of Chemical Engineering*, vol. 3, no. 6, pp. 802–809, (2011).
- [20] G. C. Wu and E. W. M. Lee, “Fractional variational iteration method and its application,” *Physics Letters A*, vol. 374, no. 25, pp. 2506–2509, (2010).
- [21] Y. Khan, N. Faraz, A. Yildirim, and Q. Wu, “Fractional variational iteration method for fractional initial-boundary value problems in the application of nonlinear science,” *Computers & Mathematics with Applications*, vol. 62, no. 5, pp. 2273–2278, (2011).
- [22] G. C. Wu, “A fractional variational iteration method for solving fractional nonlinear differential equations,” *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2186–2190, (2011).
- [23] S. Zhang and H.-Q. Zhang, “Fractional sub-equation method and its applications to nonlinear fractional PDEs,” *Physics Letters A*, vol. 375, no. 7, pp. 1069–1073, (2011).
- [24] H. Jafari, H. Tajadodi, N. Kadkhoda, and D. Baleanu, *Abstract and Applied Analysis*, Article ID 587179, (2013).
- [25] K. B. Oldham, J. Spanier, *The fractional calculus*, Academic Press, New York, (1999).
- [26] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, (2003).
- [27] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent: Part II*, *J Roy Astronom Soc.*, vol. 13, pp. 529-539, (1967).
- [28] H. Beyer, S. Kempfle, *Definition of physically consistent damping laws with fractional derivatives*, *Z. Angew Math. Mech.*, vol. 75, pp. 623-635, (1995).
- [29] W. Schneider, W. Wyss, *Fractional diffusion and wave equations*, *J. Math. Phys.*, vol. 30, pp. 134-144, (1989).
- [30] F. Mainardi, *Fractional relaxation-oscillation and fractional diffusion-wave phenomena*, *Chaos Solitons and Fractals*, vol. 7, pp. 1461-1477, (1996).
- [31] F. Huang, F. Liu, *The time fractional diffusion equation and fractional advection dispersion equation*, *ANZIAM J.* 46, pp 1-14, (2005).
- [32] J.H. He, *Nonlinear oscillation with fractional derivative and its applications*, *International Conference on Vibrating Engineering*, pp. 288–291, Dalian, China, (1998).
- [33] J.H. He, *Approximate analytical solution for seepage flow with fractional derivative in porous media*, *Comput. Methd. Appl. Mech. Eng.* 167, pp. 57-68, (1998).
- [34] N. Faraz, Y. Khan, D. S. Sankar, *Decomposition-transform method for fractional differential equations*, *J. Nonlinear Sci. Numer. Simulation*, vol. 11, pp. 305–310, (2010).
- [35] X. J. Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher, Hong Kong, (2011).
- [26] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, NY, USA, (2012).
- [37] S. Q. Wang, Y. J. Yang, and H. K. Jassim, “*Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative*,” *Abstract and Applied Analysis*, Article ID 176395, (2014).
- [38] W. P. Zhong, X. J. Yang, F. Gao, *Journal of Applied Functional Analysis*, pp. 92–99 (2013).
- [39] M. S. Hu, Ravi P. Agarwal, X. J. Yang, *Abstract Applied Analysis*, Article ID 567401 (2012).

- [40] S. P. Yan, H. Jafari, and H. K. Jassim, " Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators," *Advances in Mathematical Physics*, Article ID 161580, (2014).
- [41] C. Cattani, D. Baleanu, and X. J. Yang, "A local fractional variational iteration and Decomposition methods for Wave equation on Cantor set within local fractional Operators," *Abstract and Applied Analysis*, Article ID 535048, (2014).
- [42] Yang, X. J., Baleanu, D., Fractal Heat Conduction Problem Solved by Local Fractional Variation Iteration Method, *Thermal Science*, pp. 625-628, (2013).
- [43] Su, W. H., et al., Damped Wave Equation and Dissipative Wave Equation in Fractal Strings within the Local Fractional Variational Iteration Method, *Fixed Point Theory and Applications*, pp. 89-102, (2013).
- [44] He, J.-H., Liu, F.-J., Local Fractional Variational Iteration Method for Fractal Heat Transfer in Silk Cocoon Hierarchy, *Nonlinear Science Letters A*, pp. 15-20, (2013).
- [45] A. M. Yang, X. J. Yang, and Z. B. Li, " Local Fractional Series Expansion Method for Wave and Diffusion Equations on Cantor Sets. ," *Abstract and Applied Analysis*, Article ID 351057, (2014).